Direct Proof – Rational Numbers Lecture 14 Section 4.2

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Robb T. Koether (Hampden-Sydney College) Direct Proof – Rational Numbers

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Rational Numbers

The Harmonic Mean



3 Intersecting Lines





2 Special Rational Numbers

- 3 Intersecting Lines
- Assignment

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Definition (Rational Number)

A rational number is a real number that can be represented as the quotient of two integers. That is, a real number *r* is rational if

$$\exists a, b \in \mathbb{Z}, r = \frac{a}{b}.$$

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Theorem

The sum of two rational numbers is a rational number.

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Proof.

- Let *r* and *s* be rational numbers.
- Then there exist integers a, b, c, and d such that $r = \frac{a}{b}$ and $s = \frac{c}{d}$.
- Then

$$r + s = rac{a}{b} + rac{c}{d}$$

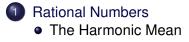
 $= rac{ad}{bd} + rac{bc}{bd}$
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Proof.

- *ad* + *bc* and *bd* are integers.
- Furthermore, $bd \neq 0$ because $b \neq 0$ and $d \neq 0$.
- Therefore, r + s is a rational number.



2 Special Rational Numbers

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Definition (Harmonic Mean)

Let x and y be two nonzero real numbers with $y \neq -x$. The harmonic mean h(x, y) of x and y is the reciprocal of the average of their reciprocals. That is,

$$h(x,y)=\frac{1}{\left(\frac{\frac{1}{x}+\frac{1}{y}}{2}\right)}.$$

Note that

$$\frac{1}{h(x,y)} = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{y}\right)$$

and that

$$h(x,y)=\frac{2xy}{x+y}.$$

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Theorem

The harmonic mean of rational numbers, when defined, is a rational number.

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Proof.

• Let *r* and *s* be nonzero rational numbers.

• Let $r = \frac{a}{b}$ and $s = \frac{c}{d}$ for some integers *a*, *b*, *c*, and *d*.

• Then *a*, *b*, *c*, and *d* are nonzero.

$$h(r,s) = \frac{1}{\left(\frac{\frac{1}{r} + \frac{1}{s}}{2}\right)} = \frac{2rs}{r+s}$$
$$= \frac{2 \cdot \frac{a}{b} \cdot \frac{c}{d}}{\frac{a}{b} + \frac{c}{d}} = \frac{2ac}{ad+bc}$$

• And $\frac{2ac}{ad+bc}$ is a rational number provided $ad+bc \neq 0$.

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Theorem

Let $0.d_1d_2d_3...$ be the decimal representation of a real number a. If the representation is repeating, that is, if there exists an integer $n \ge 1$ and an integer $m \ge 1$ such that

$$d_{n+i} = d_{n+m+i}$$

for all $i \ge 0$, then a is rational.

- How would we prove it?
- Is the converse true?





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Definition (Even Over Odd Rational)

A rational number $\frac{a}{b}$ is called even over odd if a is even and b is odd.

Theorem

Let r and s be even-over-odd rational numbers. Then r + s, r - s, and rs are also even-over-odd rational numbers.

- Is $\frac{r}{s}$ necessarily even over odd?
- Is it possible that $\frac{r}{x}$ is even over odd?

Definition (Odd Over Even Rational)

A rational number $\frac{a}{b}$ is called odd over even if a is odd and b is even.

• If *r* and *s* are odd-over-even rational numbers, then what can we say about r + s, r - s, *rs*, and $\frac{r}{s}$?



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Theorem

If two lines have distinct rational slopes and rational y-intercepts, then they intersect at a point with rational coordinates.

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Proof.

• Let the lines be

$$L_1: y = m_1 x + b_1,$$

 $L_2: y = m_2 x + b_2,$

where m_1 , m_2 , b_1 , and b_2 are rational and $m_1 \neq m_2$.

Solve the equations simultaneously to find the point of intersection.

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Proof. • We get $x=rac{b_2-b_1}{m_1-m_2},$ $y = \frac{b_1 m_2 - b_2 m_1}{m_1 - m_2}.$ Justify the claim that these are rational numbers.

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Assignment

- Read Section 4.2, pages 163 168.
- Exercises 5, 7, 8, 14, 16, 22, 27, 28, 30, 33, page 168.

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